Boson-Fermion unification implemented by Wick calculus

John Gough
Department of Computing & Mathematics
Nottingham-Trent University, Burton Street,
Nottingham NG1 4BU, United Kingdom.
john.gough@ntu.ac.uk

Abstract

We construct a transformation between Bose Fock space $\Gamma_+(\mathfrak{h})$ and Fermi Fock space $\Gamma_-(\mathfrak{h})$ that is super-symmetric in the sense that it converts Boson fields into Fermi fields over a fixed one-particle space $\mathfrak{h}$. The transformation the spectral splitting of $\mathfrak{h}$ into a continuous direct integral of internal spaces $(\mathfrak{k}_\omega)_\omega$. We present a theory of integration on the Fock spaces over $L^2\left((\mathfrak{k}_\omega)_\omega, \mathbb{R}_+, d\omega\right)$ that is a natural generalization of the theory of quantum stochastic calculus and which we refer to as a Wick calculus.

Keywords: second quantization, quantum stochastics, supersymmetry.

1 Introduction

The formalism of second quantization is fundamental to modern physics [1] and admits a natural functional calculus. A very specialized version of this is quantum stochastic calculus. Quantum stochastic processes, parameterized by time t, are families of operators on Fock space over Hilbert spaces of the type $L^2(\mathfrak{k}, \mathbb{R}_+, dt)$. That is, $\mathfrak{k}$ is a fixed Hilbert space, termed the *internal space*, and $\phi \in L^2(\mathfrak{k}, \mathbb{R}_+, dt)$ is square-integrable $\mathfrak{k}$ valued function. As is well-known, we have the natural isomorphism $L^2(\mathfrak{k}, \mathbb{R}_+, dt) \cong \mathfrak{k} \otimes L^2(\mathbb{R}_+, dt)$. The stochastic calculus has been developed in the Boson case [2], generalizing the classical Itô theory, and Fermion setting [3], generalizing the Clifford-Itô theory [4]. The Bose and Fermi theories have been unified by means of a continuous version of the Jordan-Wigner transformation [5].

Here we wish to consider second quantizations of Hilbert spaces of the type $L^2\left((\mathfrak{k}_\omega)_\omega, \mathbb{R}_+, d\omega\right)$ where now we work with families $(\mathfrak{k}_\omega)_\omega$ of internal spaces indexed by parameter(interpreted here as frequency) $\omega > 0$. The study of Wiener-Itô integrals (in the time domain) on Fock spaces over direct integral Hilbert spaces was first considered by Sunder [6].

Our motivation comes from modelling physical quantum reservoirs. In such cases, the $\mathfrak{k}_\omega$ arise as the mass shell Hilbert spaces for a fixed energy ω.

An infinitely extended quantum reservoir can be considered as the second quantization of particle having one-particle space $\mathfrak{h}$ and having one-particle Hamiltonian $H \geq 0$ which is a fixed self-adjoint operator H on $\mathfrak{h}$. Specifically, we take H to have absolutely continuous spectrum. There then exists an orthogonal projection valued measure $\Pi[.]$ concentrated on $[0,\infty)$ such that

$$H \equiv \int_{[0,\infty)} \omega \, \Pi[d\omega]. \tag{1.1}$$

Given $\Omega > 0$ we consider the subspace $\mathcal{D}_\Omega \subset \mathfrak{h}$ such that

$$\int_{-\infty}^{+\infty} dt \ \left| \langle f | \exp\{i(H-\Omega)t\} g\rangle \right| < \infty \tag{1.2}$$

whenever $f, g \in \mathcal{D}_\Omega$. On this domain we define the sesquilinear form

$$(g|f)_\Omega := \int_{-\infty}^{+\infty} \langle f | \exp\{i(H-\Omega)t\} g\rangle \ dt \tag{1.3}$$

and it is convenient to consider the Hilbert space $\mathfrak{k}_\Omega$ obtained by factoring out from $\mathcal{D}_\Omega$ the null elements $\mathcal{N}_\Omega = \{f : (f|f)_\Omega = 0\}$ and taking the Hilbert space completion with respect to the $(.|.)_\Omega$-norm. Formally we have

$$2\pi \ \langle f | \Pi[d\omega] \ g\rangle \equiv (g|f)_\omega \ d\omega \tag{1.4}$$

We then obtain the continuous direct integral decomposition

$$\mathfrak{h} \cong \int_{[0,\infty)}^{\oplus} d\omega \ \mathfrak{k}_\omega. \tag{1.5}$$

For each $\omega \geq 0$, $\mathfrak{k}_\omega$ is a Hilbert space with inner product $(.|.)_\omega$ and we may consider $\mathfrak{h}$ to consists of all vectors $\phi = (\phi_\omega)_{\omega \geq 0}$, where $\phi_\omega \in \mathfrak{k}_\omega$ and $\int_{[0,\infty)} (\phi_\omega|\phi_\omega)_\omega < +\infty$. The inner product on $\mathfrak{h}$ may be represented by

$$\langle \phi | \psi \rangle = \int_{[0,\infty)} d\omega \ (\phi_\omega|\psi_\omega)_\omega. \tag{1.6}$$

We may write $\mathfrak{h}$ as $L^2\left((\mathfrak{k}_\omega)_\omega, \mathbb{R}_+, d\omega\right)$. It is natural, in the light of the development of quantum stochastic calculus, to consider the space $L^2_{\mathrm{loc}}\left((\mathfrak{k}_\omega)_\omega, \mathbb{R}_+, d\omega\right)$ of locally square-integrable objects $\phi = (\phi_\omega)_\omega$ where now we only require $\int_B d\omega \ (\phi_\omega|\phi_\omega)_\omega < +\infty$ for any compact Borel subset B.

Example: The basic situation we have in mind [9] is a reservoir particle moving in ν-dimensions and having the spectrum of elementary excitations $\omega = \omega(\mathbf{k})$ where $\mathbf{k} = (k_1, \dots, k_\nu)$ are the momenta coordinate. We take $\mathfrak{h} = L^2(\mathbb{R}^\nu, d^\nu k)$ and $(Hf)(\mathbf{k}) = \omega(\mathbf{k}) f(\mathbf{k})$. We shall take it that the spectrum foliates the momentum space into the mass shells $M_\omega := \{\mathbf{k} : \omega(\mathbf{k}) = \omega\}$ and that the

Lesbegue measure on $\mathbb{R}^\nu$ can be decomposed locally as $d^\nu k = d\omega \times d\sigma_\omega$ where $\omega = \omega(\mathbf{k})$ and $d\sigma_\omega$ is surface measure on M_ω. (For instance, if H corresponds to $-\Delta$ in the position representation then $\sqrt{\omega(\mathbf{k})}$ is radial coordinate and $d\sigma_\omega$ will be surface measure on the sphere of radius $\sqrt{\omega}$.) The individual Hilbert spaces are $\mathfrak{k}_\omega = L^2(M_\omega, d\sigma_\omega)$.

Note that we can deal with quasi-free gauge-invariant states on $\Gamma_\pm(\mathfrak{h})$ provided only that the covariance matrix Q commutes with the filtration of $\mathfrak{h}$ induced by H. Since we have $Q = \coth\frac{\beta H - \mu}{2}$ (Bose) and $Q = \tanh\frac{\beta H - \mu}{2}$ (Fermi) for the free particle Gibbs states, this construction arises naturally.

2 Mathematical Notations and Preliminaries

Let $\Gamma(\mathfrak{h}) := \bigoplus_{n=0}^\infty \mathfrak{h}^{\otimes n}$ be the "Full" Fock space over a fixed complex separable Hilbert space $\mathfrak{h}$. The (anti)-symmetrization operators $\Pi_\pm$ are define through linear extension of the relations $\Pi_\pm f_1 \otimes \cdots \otimes f_n := \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} (\pm)^\sigma \ f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)}$, with $f_j \in \mathfrak{h}$, $\mathfrak{S}_n$ denotes the permutation group on $\{1,\dots,n\}$ and $(-1)^\sigma$ is the parity of the permutation σ. The Bose Fock space $\Gamma_+(\mathfrak{h})$ and the Fermi Fock space $\Gamma_-(\mathfrak{h})$ having $\mathfrak{h}$ as one-particle space are then defined as the subspaces $\Gamma_\pm(\mathfrak{h}) := \Pi_\pm \Gamma(\mathfrak{h})$. As usual, we distinguish the Fock vacuum $\Phi = (1, 0, 0\dots)$, though we shall write $\Phi_\pm$ for emphasis.

Let $g \in \mathfrak{h}$, U unitary and H self-adjoint on $\mathfrak{h}$. We define the following operators on the Full Fock space

$$\begin{aligned} A^+(h)\ f_1\otimes\cdots\otimes f_n \quad &: \quad = \sqrt{n+1}\, h\otimes f_1\otimes\cdots\otimes f_n; \\ A^-(h)\ f_1\otimes\cdots\otimes f_n \quad &: \quad = \frac{1}{\sqrt{n}}\ \langle h|f_1\rangle\ f_2\otimes\cdots\otimes f_n; \\ \Gamma(U)\ f_1\otimes\cdots\otimes f_n \quad &: \quad = (Uf_1)\otimes\cdots\otimes(Uf_n); \\ \gamma(H)\ f_1\otimes\cdots\otimes f_n \quad &: \quad = \sum_j f_1\otimes\cdots\otimes(Hf_j)\otimes\cdots\otimes f_n. \end{aligned} \tag{2.1}$$

Bose creation and annihilation fields are then defined on $\Gamma_+(\mathfrak{h})$ as

$$B^\pm(h) := \Pi_+ A^\pm(h)\,\Pi_+ \tag{2.2}$$

while Fermi creation and annihilation fields are defined on $\Gamma_+(\mathfrak{h})$ as

$$F^\pm(h) := \Pi_- A^\pm(h)\,\Pi_-. \tag{2.3}$$

Using the traditional conventions $[A,B] = AB - BA$ and $\{A,B\} = AB + BA$, we have the canonical (anti)-commutation relations

$$\left[B^-(f), B^+(g)\right] = \langle f|g\rangle; \quad \left\{F^-(f), F^+(g)\right\} = \langle f|g\rangle. \tag{2.4}$$

Second quantization operators are defined as $\Gamma_\pm(U) := \Pi_\pm \Gamma(U)\,\Pi_\pm$ and differential second quantization operators as $\gamma_\pm(U) := \Pi_\pm\,\gamma(U)\,\Pi_\pm$. We have the relation

$$\exp\left\{it\,\gamma_\pm(H)\right\} = \Gamma_\pm\left(e^{itH}\right). \tag{2.5}$$

More generally we may take the argument of the differential second quantizations to be bounded: for the rank-one operator $H = |f\rangle\langle g|$ described in standard Dirac bra-ket notation, we have $\gamma_+(|f\rangle\langle g|) \equiv B^+(f)\,B^-(g)$ and $\gamma_-(|f\rangle\langle g|) \equiv F^+(f)\,F^-(g)$.The following relations will be useful

$$\Gamma_+(U)\ B^\pm(\phi)\ \Gamma_+\left(U^\dagger\right) = B^\pm(U\phi)\,; \quad \Gamma_-(U)\ F^\pm(\phi)\ \Gamma_-\left(U^\dagger\right) = F^\pm(U\phi) \tag{2.6}$$

In the Bose case, the exponential vector map $\varepsilon : \mathfrak{h} \mapsto \Gamma_+(\mathfrak{h})$ is introduced as

$$\varepsilon(f) = \oplus_{n=0}^\infty \frac{1}{\sqrt{n!}} f^{\otimes n} \tag{2.7}$$

with $f^{\otimes n}$ the n-fold tensor product of f with itself. The Fock vacuum is, in particular, given by $\Phi = \varepsilon(0)$. The set $\varepsilon(\mathfrak{h})$ is total in $\Gamma_+(\mathfrak{h})$ and we note that $\langle \varepsilon(f)\,|\varepsilon(g)\rangle = \exp\langle f|g\rangle$. The next result is the basis for the development of a calculus of second quantized fields, see [7] for proofs.

Lemma (2.1): *The operations of Bose or Fermi second quantization have the natural functorial property*

$$\Gamma_\pm(\mathfrak{h}_1 \oplus \mathfrak{h}_2) \cong \Gamma_\pm(\mathfrak{h}_1) \otimes \Gamma_\pm(\mathfrak{h}_2)\ . \tag{2.8}$$

3 Spectral Processes and Wick Calculus

Let $\mathfrak{h}$ be a separable complex Hilbert space admitting the continuous direct integral decomposition

$$\mathfrak{h} = \int_{[0,\infty)}^\oplus d\omega\ \mathfrak{k}_\omega. \tag{3.1}$$

That is, for each $\omega \geq 0$, $\mathfrak{k}_\omega$ is a Hilbert space with inner product $(.|.)_\omega$ and we have that $\mathfrak{h}$ consists of all vectors $\phi = (\phi_\omega)_{\omega\geq 0}$, where $\phi_\omega \in \mathfrak{k}_\omega$ and we have the inner product on $\mathfrak{h}$

$$\langle\phi|\psi\rangle = \int_{[0,\infty)} d\omega\ \left(\phi_\omega|\psi_\omega\right)_\omega . \tag{3.2}$$

For $0 \leq \Omega_1 < \Omega_2$, let $\mathfrak{k}_{[\Omega_1,\Omega_2]} = \int_{[\Omega_1,\Omega_2]}^\oplus d\omega\ \mathfrak{k}_\omega$, then in these notations

$$\mathfrak{h} \cong \mathfrak{k}_{[0,\Omega]} \oplus \mathfrak{k}_{(\Omega,\infty)} \tag{3.3}$$

for each $\Omega > 0$. This leads to the following continuous tensor product decomposition for Fock space

$$\Gamma_+(\mathfrak{h}) \cong \Gamma_+\left(\mathfrak{k}_{[0,\Omega]}\right) \otimes \Gamma_+\left(\mathfrak{k}_{(\Omega,\infty)}\right). \tag{3.4}$$

We define an absolutely continuous, orthogonal projection valued measure Π on $[0,\infty)$ by

$$\left(\Pi_A\phi\right)_\omega := 1_A(\omega)\,\phi_\omega \tag{3.5}$$

where A is any Borel set and 1_A its characteristic function.

Next of all, fix an *initial Hilbert space* $\mathfrak{H}_0$ and set

$$\mathfrak{H} := \mathfrak{H}_0 \otimes \Gamma_+ (\mathfrak{h}) ; \quad \mathfrak{H}_{\Omega]} := \mathfrak{H}_0 \otimes \Gamma_+ \left(\mathfrak{k}_{[0,\Omega]} \right) ; \quad \mathfrak{H}_{(\Omega} := \Gamma_+ \left(\mathfrak{k}_{(\Omega,\infty)} \right) . \tag{3.6}$$

Definition (3.1): *A family $(X_\Omega)_{\Omega \geq 0}$ of operators on $\mathfrak{H}$ is said to be spectrally-adapted if, for each $\Omega > 0$, the operator X_Ω is the algebraic ampliation to $\mathfrak{H}_0 \underline{\otimes} \varepsilon \left(\mathfrak{k}_{[0,\Omega)} \right) \underline{\otimes} \varepsilon \left(\mathfrak{k}_{(\Omega,\infty)} \right)$ of an operator on $\mathfrak{H}_{\Omega]}$ with domain $\mathfrak{H}_0 \underline{\otimes} \varepsilon \left(\mathfrak{k}_{[0,\Omega)} \right)$. We also demand the existence of an adjoint process $\left(X_\Omega^\dagger \right)_{\Omega \geq 0}$ having the same ampliation structure. (Here $\underline{\otimes}$ denotes the algebraic tensor product.)*

Definition (3.2): *Let $\phi \in \mathfrak{h}$, we define the (Bosonic) creation and annihilation spectral processes on $\Gamma_+ (\mathfrak{h})$ to be*

$$B_\phi^\pm (\Omega) := B^\pm \left(\Pi_{[0,\Omega]} \phi \right) \tag{3.7}$$

and the conservation spectral process to be

$$\Lambda (\Omega) := \gamma_+ \left(\Pi_{[0,\Omega]} \right) \tag{3.8}$$

for each $\Omega \geq 0$.

The operators $B_\phi^\pm (\Omega)$, $\Lambda (\Omega)$ are spectrally-adapted in this sense.

Let $(X_{jk} (\Omega))_{\Omega \geq 0}$ be piecewise constant, spectrally-adapted processes for $j, k \in \{0, 1\}$. The *Wick integral*

$$X_\Omega = \int_{[0,\Omega]} \Big(X_{11} \otimes \Lambda (d\omega) + X_{10} \otimes B_\phi^+ (d\omega) + X_{01} \otimes B_\psi^- (d\omega) + X_{00} \otimes d\omega \Big) \tag{3.9}$$

is defined in such a way that $\langle u \otimes \varepsilon (f) \,|\, X \, v \otimes \varepsilon (g) \rangle$ is interpreted as

$$\int_0^\Omega \langle u \otimes \varepsilon (f) \,| \, \left(X_{11} \left(f_\omega | g_\omega \right)_\omega + X_{10} \left(f_\omega | \phi_\omega \right)_\omega + X_{01} \left(\psi_\omega | g_\omega \right)_\omega + X_{00} \right) \, v \otimes \varepsilon (g) \rangle \, d\omega \tag{3.10}$$

for all $u, v \in \mathfrak{H}_0$ and $f, g \in \mathfrak{h}$. Formally we write this as $X_\Omega \equiv \int_{[0,\Omega]} X (d\omega)$. Similarly, we set $X_\Omega^\dagger = \int_{[0,\Omega]} (X_{11} \otimes \Lambda (d\omega) \; + X_{10} \otimes B_\phi^- (d\omega) \; + X_{01} \otimes B_\psi^+ (d\omega) \; + X_{00} \otimes d\omega)$.

Lemma (3.3): *Let X_Ω be the stochastic integral with piecewise constant, spectrally adapted coefficients as above, then*

$$\begin{aligned} \left\| X_\Omega \, u \otimes \varepsilon (f) \right\|^2 \leq & \int_{[0,\Omega]} d\omega \, \exp \left\{ \Omega - \omega + 3 \int_\omega^\Omega \left(f_{\omega'} | f_{\omega'} \right) d\omega' \right\} \\ & \times \Big[3 \left(f_\omega | f_\omega \right)_\omega \left\| X_{11} (\omega) \; u \otimes \varepsilon (f) \right\|^2 + 3 \left(\phi_\omega | \phi_\omega \right)_\omega \left\| X_{10} (\omega) \; u \otimes \varepsilon (f) \right\|^2 \\ & + \left(\psi_\omega | \psi_\omega \right)_\omega \left\| X_{01} (\omega) \; u \otimes \varepsilon (f) \right\|^2 + \left\| X_{00} (\omega) \; u \otimes \varepsilon (f) \right\|^2 \Big] . \end{aligned} \tag{3.11}$$

Proof. This is a generic type of estimate in quantum stochastic calculus and in our case is a straightforward adaptation of section 2 of [8] and we omit it. ∎

Let $(X_{jk}(\Omega))_{\Omega\geq 0}$ be spectrally adapted processes that are weakly measurable and satisfy the following locally square-integrability conditions (for arbitrary $u \in \mathfrak{H}_0,\ f \in \mathfrak{h}$)

$$\int_{[0,\Omega]} d\omega \ \left(f_\omega|f_\omega\right)_\omega \left\| X_{11}(\omega)\ u\otimes\varepsilon(f)\right\|^2 \quad < \quad \infty;$$

$$\int_{[0,\Omega]} d\omega \ \left\| X_{jk}(\omega)\ u\otimes\varepsilon(f)\right\|^2 \quad < \quad \infty, \qquad \text{otherwise.}$$

Then the Wick integral

$$X_\Omega = \int_{[0,\Omega]} \Big(X_{11}\otimes\Lambda(d\omega) + X_{10}\otimes B_\phi^+(d\omega) + X_{01}\otimes B_\psi^-(d\omega) + X_{00}\otimes d\omega \Big)$$

exists and is well-defined. It can be understood as the limit of an approximating sequence $\left(X_\Omega^{(n)}\right)_{\Omega\geq 0}$, each one constructed using piecewise continuous coefficients: the approximating coefficients $X_j^{(n)}$ should be chosen so that $\int_{[0,\Omega]} d\omega \ \left\|\left(X_{jk}-X_{jk}^{(n)}\right)\right\|^2 \to 0$ and the limit process will be independent of the approximating sequence used.

It is useful to use the differential notation $X_\Omega \equiv \int_{[0,\Omega]} X(d\omega)$ with $X(d\omega) = X_{11}\otimes\Lambda(d\omega)\ +X_{10}\otimes B_\phi^+(d\omega)\ +X_{01}\otimes B_\psi^-(d\omega)\ +X_{00}\otimes d\omega$. We can readily obtain the integral relation

$$\begin{aligned}
&\left\langle u\otimes\varepsilon(f)\,|\left[X_\Omega Y_\Omega - X_0 Y_0 - \int_0^\Omega X_\omega Y(d\omega) - \int_0^\Omega X(d\omega)\,Y_\omega\right] v\otimes\varepsilon(g)\right\rangle \\
= \ & \int_{[0,\Omega]} \langle u\otimes\varepsilon(f)\,|\,[X_{11}Y_{11}\left(f_\omega|g_\omega\right)_\omega + X_{11}Y_{10}\left(f_\omega|\phi_\omega\right)_\omega] \\
& +X_{01}Y_{11}\left(\psi_\omega|g_\omega\right)_\omega + X_{01}Y_{10}]\ v\otimes\varepsilon(g)\rangle\ d\omega
\end{aligned} \qquad (3.12)$$

Here we encounter a familiar problem from quantum mechanics: the product of Wick ordered expressions is not immediately Wick ordered. The non-Leibniz term in (3.12) is the result of putting to Wick order, in quantum stochastic calculus it would be called the Itô correction, and for bounded coefficients we have the Itô product formula

$$(XY)(d\omega) = X(d\omega)\,Y(\omega) + X(\omega)\,Y(d\omega) + X(d\omega)\,Y(d\omega) \qquad (3.13)$$

which is evaluated by the rule that the fundamental differentials $\Lambda(d\omega), B_\phi^\pm(d\omega)$ and $d\omega$ commute with spectrally adapted processes and by the quantum spectral

Itô table:

$$\begin{array}{l|llll}
 & \Lambda\left(d\omega\right) & B_{\phi}^{+}\left(d\omega\right) & B_{\phi}^{-}\left(d\omega\right) & d\omega \\ \hline
\Lambda\left(d\omega\right) & \Lambda\left(d\omega\right) & B_{\phi}^{+}\left(d\omega\right) & 0 & 0 \\
B_{\psi}^{+}\left(d\omega\right) & 0 & 0 & 0 & 0 \\
B_{\psi}^{-}\left(d\omega\right) & B_{\psi}^{-}\left(d\omega\right) & \left(\psi_{\omega}|\phi_{\omega}\right)_{\omega}\, d\omega & 0 & 0 \\
d\omega & 0 & 0 & 0 & 0
\end{array} \tag{3.14}$$

4 Super-symmetric Spectral Transformations

Definition (4.1): *We define the spectral parity processes to be*

$$J_{\Omega} := \Gamma_{+}\left(-\Pi_{[0,\Omega]} + \Pi_{(\Omega,\infty)}\right). \tag{4.1}$$

With respect to the decomposition (3.4) we have $J_{\Omega} \equiv (-1)^{\Lambda(\Omega)} \otimes 1_{(\Omega}$.

Lemma (4.2): *The process $(J_{\Omega})_{\Omega\geq 0}$ is a unitary, self-adjoint, frequency-adapted process satisfying the properties*

1. $[J_{\omega}, J_{\omega'}] = 0$;
2. $J_{\omega}\Phi_{+} = \Phi_{+}$;
3. $dJ_{\omega} = -2J_{\omega} \otimes \Lambda\left(d\omega\right), J_{0} = 1$.

Proof. Property 1 follows is immediate from the observation that $J_{\omega}J_{\omega'} = \Gamma_{+}\left(\Pi_{[0,a)} - \Pi_{[a,b]} + \Pi_{(b,\infty)}\right)$ where $a = \omega \wedge \omega'$ and $b = \omega \vee \omega'$. Property 2 is evident from the fact that $\Phi_{+} = \varepsilon\left(0\right)$.

Next from the rule $\Lambda\left(d\omega\right)\Lambda\left(d\omega\right) = \Lambda\left(d\omega\right)$, we have

$$df\left(\Lambda\left(\omega\right)\right) = \left[f\left(\Lambda\left(\Omega\right) + 1\right) - f\left(\Lambda\left(\omega\right)\right)\right] \otimes \Lambda\left(d\omega\right)$$

for analytic functions f. Setting $f\left(\lambda\right) = \exp\left\{i\pi\lambda\right\}$ gives property 3. ■

Definition (4.3): *Let $\phi \in \mathfrak{h}$, we define the Fermionic creation and annihilation spectral processes to be*

$$F_{\phi}^{\pm}\left(\Omega\right) := \int_{[0,\Omega]} J_{\omega} \otimes B_{\phi}^{\pm}\left(d\omega\right). \tag{4.2}$$

Lemma (4.4): *For each $\Omega \geq 0$ the Fermionic processes $F_{\phi}^{\pm}\left(\Omega\right)$ anti-commute with the parity operator J_{Ω}.*

Proof. We first note that the exponential vectors are a stable domain for the parity operator and the Bosonic, and hence Fermionic, processes. In particular

we have

$$\left\langle \varepsilon(f) \mid \left\{ J_\Omega, F_\phi^-(\Omega) \right\} \varepsilon(g) \right\rangle = \left\langle \varepsilon(f) \mid J_\Omega F_\phi^-(\Omega)\, \varepsilon(g) \right\rangle + \left\langle \varepsilon(f) \mid F_\phi^-(\Omega)\, J_\Omega\, \varepsilon(g) \right\rangle$$

$$= \int_{[0,\Omega]} d\omega\ (\phi_\omega | g_\omega)_\omega\ \left\langle \varepsilon\left(-\Pi_{[0,\Omega]} f + \Pi_{(\Omega,\infty)} f\right) \mid \varepsilon\left(-\Pi_{[0,\omega]} g + \Pi_{(\omega,\infty)} g\right) \right\rangle$$

$$- \int_{[0,\Omega]} d\omega\ (\phi_\omega | g_\omega)_\omega\ \left\langle \varepsilon(f) \mid \varepsilon\left(-\Pi_{[0,\Omega]} g + \Pi_{(\Omega,\infty)} g\right) \right\rangle$$

$$= \int_{[0,\Omega]} d\omega\ (\phi_\omega | g_\omega)_\omega\ \langle \varepsilon(f) \mid (J_\Omega J_\omega - J_\omega J_\Omega)\ \varepsilon(g) \rangle = 0$$

and so we deduce that $\left\{ J_\Omega, F_\phi^-(\Omega) \right\} = 0$. The proof of the relation $\left\{ J_\Omega, F_\phi^+(\Omega) \right\}$ $= 0$ is similar. ■

Theorem (4.5): *The Fermionic processes $F_\phi^\pm(\Omega)$ are bounded and satisfy the canonical anti-commutation relations*

$$\left\{ F_\phi^-(\Omega), F_\psi^+(\Omega) \right\} = \int_{[0,\Omega]} d\omega\ (\phi_\omega | \psi_\omega)_\omega\,; \tag{4.3a}$$

$$\left\{ F_\phi^-(\Omega), F_\psi^-(\Omega) \right\} = 0 = \left\{ F_\phi^+(\Omega), F_\psi^+(\Omega) \right\}. \tag{4.3b}$$

Proof. Using quantum spectral calculus we have

$$\begin{aligned} d\left\{ F_\phi^-(\omega), F_\psi^+(\omega) \right\} &= \left\{ J_\omega, F_\psi^+(\omega) \right\} \otimes B_\phi^-(d\omega) + \left\{ F_\phi^-(\omega), J_\omega \right\} \otimes B_\psi^+(d\omega) \\ &\quad + J_\omega^2 \otimes B_\phi^-(d\omega)\, B_\psi^+(d\omega) \\ &= (\phi_\omega | \psi_\omega)_\omega\, d\omega \end{aligned}$$

which can be integrated to obtain (4.3a).

Next, since $dF_\phi^-(\omega) = J_\omega \otimes B_\phi^+(d\omega)$, the Itô formula (3.13) gives

$$\left(dF_\phi^-(\omega) \right)^2 = \left(J_\omega F_\phi^-(\omega) + F_\phi^-(\omega)\, J_\omega \right) \otimes B_\phi^-(d\omega) = 0$$

again by virtue of the previous lemma. Since $F_\phi^-(0) = 0$, the equation can be integrated to show that $F_\phi^-(\omega)^2 = 0$. Likewise, we can show that $F_\phi^+(\omega)^2 = 0$. We then obtain (4.3b) through polarization. ■

Lemma (4.6): *For each $\Omega \geq 0$ and $\phi_1, \ldots, \phi_n \in \mathfrak{h}$ we have the identity*

$$F_{\phi_1}^+(\Omega) \cdots F_{\phi_n}^+(\Omega)\ \Phi_+ = \sum_{\sigma \in \mathfrak{S}_n} (-1)^\sigma \int_{0<\omega_1<\cdots\omega_n<\Omega} B_{\phi_{\sigma(1)}}^+(d\omega_1) \cdots B_{\phi_{\sigma(n)}}^+(d\omega_n)\ \Phi_+$$

where $(-1)^\sigma$ is the parity of the permutation $\sigma \in \mathfrak{S}_n$.

Proof. The proof will be one by induction. The identity is trivially true for $n=1$.The rules of the spectral quantum stochastic calculus yields

$$d\left[F_{\phi_1}^{+}(\omega)\cdots F_{\phi_{n+1}}^{+}(\omega)\right]=$$

$$\sum_{k=1}^{n+1}(-1)^{n-k}\left[F_{\phi_1}^{+}(\omega)\cdots\widehat{F_{\phi_k}^{+}(\omega)}\cdots F_{\phi_{n+1}}^{+}(\omega)\,J_\omega\right]\otimes B_{\phi_k}^{+}(d\omega)$$

therefore

$$\left\langle\varepsilon(f)\,|\,F_{\phi_1}^{+}(\Omega)\cdots F_{\phi_n}^{+}(\Omega)\;\Phi_+\right\rangle=$$

$$\int_0^\Omega d\omega\sum_{k=1}^{n+1}(-1)^{n-k}\left\langle\varepsilon(f)\,|\,F_{\phi_1}^{+}(\omega)\cdots\widehat{F_{\phi_k}^{+}(\omega)}\cdots F_{\phi_{n+1}}^{+}(\omega)\;\Phi_+\right\rangle(f_\omega|\phi_k(\omega))_\omega\,;$$

therefore, if the identity holds up to the n-th order, then it holds for $n+1$ also. ■

Corollary (4.7): *For each $\Omega\geq 0$ and $\phi_1,\ldots,\phi_n\in\mathfrak{h}$ we have the identity*

$$B_{\phi_1}^{+}(\Omega)\cdots B_{\phi_n}^{+}(\Omega)\;\Phi_+=\sum_{\sigma\in\mathfrak{S}_n}\int_{0<\omega_1<\cdots\omega_n<\Omega}F_{\phi_{\sigma(1)}}^{+}(d\omega_1)\cdots F_{\phi_{\sigma(n)}}^{+}(d\omega_n)\;\Phi_+.$$

Theorem (4.8): *There exists a unique unitary mapping $\Xi:\Gamma_+(\mathfrak{h})\mapsto\Gamma_-(\mathfrak{h})$ with the properties*

1. $\Xi\,\Phi_+=\Phi_-$;
2. $\Xi\,F_\phi^{\pm}(\Omega)\;\Xi^{-1}=F^{\pm}\left(\Pi_{[0,\Omega]}\phi\right)$.

Proof. First of all, observe that any n-particle vector $\Pi_+f_1\otimes\cdots\otimes f_n$ can be written as $B^+(f_1)\cdots B^+(f_n)\,\Phi_+$ and so any vector in $\Gamma_+(\mathfrak{h})$ can be obtained a sums of products of creators acting on the Fock vacuum. In other words Φ_+ is cyclic in $\Gamma_+(\mathfrak{h})$ for the $B^+(.)$ fields. Likewise, Φ_+ is cyclic for the Fermionic creator fields $F_\phi^{+}(\Omega)$ too.

We next of all note that the Fermionic annihilator fields $F_\phi^{\pm}(\Omega)$ annihilate Φ_+. We consider the mapping $\Xi:\Gamma_+(\mathfrak{h})\mapsto\Gamma_-(\mathfrak{h})$ defined by linear extension from $\Xi(\Phi_+)=\Phi_-$ and $\Xi\left(F_{\phi_1}^{+}(\Omega)\ldots F_{\phi_n}^{+}(\Omega)\,\Phi\right)=\Pi_-\left(\Pi_{[0,\Omega]}\phi_1\right)\otimes\cdots\otimes\left(\Pi_{[0,\Omega]}\phi_n\right)$.

It is readily seen that Ξ is a densely defined isometry and so extends to a unitary. ■

Theorem (4.9): *The unitary mapping $\Xi:\Gamma_+(\mathfrak{h})\mapsto\Gamma_-(\mathfrak{h})$ has the following covariance for the differential second quantizations*

$$\Xi\,\gamma_+\left(\Pi_{[0,\Omega]}\right)\,\Xi^{-1}=\gamma_-\left(\Pi_{[0,\Omega]}\right).\tag{4.4}$$

Proof. Actually, (4.4) is the differential version of the relation $\Xi\,\Gamma_+\left(U_t\right)\,\Xi^{-1} = \gamma_-\left(U_t\right)$ where U_t is the unitary $\exp\left\{it\Pi_{[0,\Omega]}\right\} = e^{it}\Pi_{[0,\Omega]} + \Pi_{(\Omega,\infty)}$. This will follow from the fact that $\Gamma_+\left(U_t\right)\,F^{\pm}\left(\phi,\Omega\right)\,\Gamma_+\left(U_t^{\dagger}\right) = F^{\pm}\left(U_t\phi,\Omega\right)$ and this is established on the domain of exponential vectors:

$$
\begin{aligned}
&\left\langle \varepsilon\left(f\right) | \,\Gamma_+\left(U_t\right)\,F_\phi^{\pm}\left(\Omega\right)\,\Gamma_+\left(U_t^{\dagger}\right)\,\varepsilon\left(g\right)\right\rangle = \left\langle \varepsilon\left(U_t^{\dagger}f\right) | \,F_\phi^{\pm}\left(\Omega\right)\,\varepsilon\left(U_t^{\dagger}g\right)\right\rangle \\
= \quad & \int_0^\Omega d\omega\,\left\langle \varepsilon\left(U_t^{\dagger}f\right) | \,J_\omega \otimes B_\phi^{\pm}\left(d\omega\right)\,\varepsilon\left(U_t^{\dagger}g\right)\right\rangle \\
= \quad & \left\langle \varepsilon\left(f\right) | \,F_{U_t\phi}^{\pm}\left(\Omega\right)\,\varepsilon\left(g\right)\right\rangle
\end{aligned}
$$

and we stress the importance of the fact that U_t is diagonal in our spectral decomposition. ■

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